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IR renormalization of general effective actions and Hawking flux in 2D gravity theories

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Abstract. The infrared problem of the effective action in 2D is discussed in the framework of the covariant perturbation theory. The divergences are regularized by a mass and the leading term is evaluated up to the third order of perturbation theory. A summation scheme is proposed which isolates the divergences from the finite part of the series and results in a single term. The latter turns out to be equivalent to the coupling to a certain classical external field. This suggests renormalization by factorization.

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1 Introduction

In the last years two-dimensional models have widely been used in the context of field quantization of and upon curved spacetimes [1]. By choosing an appropriate "potential" for the dilaton fields in the first order version of such gravity models, essentially all physically interesting theories can be covered, including interactions of gravity with matter, at least as long as no explicit couplings of the matter (e.g. scalar fields) to the scalar curvature are assumed. In particular, spherically reduced gravity (SRG) provides a simple model to study four-dimensional Hawking radiation by a two-dimensional dilaton action with just one dilaton field [2–4]. This is of special importance in connection with the correct computation of the Hawking flux, which had been the subject of some controversy [5, 6]. It has been settled by one of the present authors with D.V. Vassilevich [3], who showed that the correct flux at infinity, resulting in the Stefan–Boltzmann law as determined by the Hawking temperature, can be obtained. On the other hand, a logarithmic divergence (cf. also [7, 8]) of the flux at the horizon (in global coordinates) seems to remain a difficulty, at least for the fixed background of an "eternal" black hole[1].

The key problem has always been the computation of the effective action, when the path integral of the quantum field S (we only consider a single scalar field) is carried out. In previous work cited above [3] this was treated by the heat-kernel method, but at one point applied beyond its mathematically established range of applicability.

Another quite different approach uses the covariant perturbation theory (CPTH) of the functional determinant invented by Barvinsky and Vilkovisky [11–13]. We have shown recently that even for massless scalar fields the expected result [3] for the Hawking flux could be reproduced correctly using the CPTH in two dimensions [2]. The drawback therein, however, has been the appearance of infrared (IR) divergences.

The IR problem in two dimensions has been a matter of confusion since the invention of CPTH. The authors of that method themselves claimed the non-analyticity and thus non-applicability of their method in two dimensions, except for one particular case (namely vanishing endomorphism; see below) in which the major result, the trace anomaly, could be derived by the local Seeley–DeWitt expansion as well [14, 15]. On the other hand, in other work [16] the IR problem also has been declared as non-existing[2]. Our viewpoint is that IR divergences exist to all orders of CPTH, however, a procedure can be given to regularize and eventually renormalize them by physical arguments.

In the present paper a regularization procedure by a mass term m^2 for general effective actions in two dimensions is proposed. We consider divergent terms $\ln m^2$ up to the third order of CPTH and conjecture the possibility of a summation of the series into a single term. Finally, we show how such a term can be produced by an ambiguity of a source term coupled to the scalar field S, representing some external classical field.

In Sect. 2 we briefly recall the main features of CPTH in two dimensions and quote some specifications when applied to SRG. The subsequent Sect. 3 is devoted to a formal

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[1] A recent attempt to also include the evaporation process in a semi-classical manner can be found in [9, 10].

[2] The origin of this surprising result, which seems to contradict the analysis of Barvinsky and Vilkovisky, might be that in their case orders in the (dimensionless!) dilaton instead of orders in the curvature were considered.

analysis of the divergences. Section 4 contains the results of the mass regularization of the effective action up to the third order of CPTH. A possible summation of the divergences is conjectured. In Sect. 5 we discuss the significance of the closed divergent expression obtained in Sect. 4 and propose a renormalization by an external field.

The results are summarized in the Conclusions, where related aspects and possible extensions are discussed as well.

The Appendices contain details of the calculations on which this work is based. Appendix A presents the mass regularization for the second order of CPTH, Appendix B for the third order. The meaning of formal terms of type $\ln\Box$ is discussed in Appendix C.

2 Effective action in CPTH

The effective action W for a scalar field S on a curved spacetime L with metric g is defined by

$$\mathrm{e}^{\mathrm{i}W} = \mathcal{N}\int \mathcal{D}\left(\sqrt[4]{-g}\tilde{S}\right)\mathrm{e}^{-\frac{\mathrm{i}}{2}\int_L \tilde{S}\mathcal{O}\tilde{S}\sqrt{-g}\mathrm{d}^2x}\,, \tag{1}$$

where the factor $\sqrt[4]{-g}$ in the path-integral measure has been introduced to preserve general covariance [17], but can be eliminated right away by redefining $S := \sqrt[4]{-g}\tilde{S}$. The classical scalar field S obeys the equation of motion $\mathcal{O}S = 0$ with general d'Alembertian

$$\mathcal{O} = \Box + E\,. \tag{2}$$

E (for endomorphism) refers to some potential coupled to S which may contain one or several (dilaton) fields. It shall not depend on S, excluding self-interaction. In SRG we have only one dilaton field ϕ, defined by $X = \mathrm{e}^{-2\phi}$, where X may be gauged to simply represent the radius coordinate r^2 of the two-sphere in four dimensions. The corresponding endomorphism, expressed by ϕ, reads [2]

$$E = \Box\phi - (\nabla\phi)^2. \tag{3}$$

After introduction of Euclidean time $\tau = \mathrm{i}t$ and an Euclidean operator $\mathcal{O}_\mathcal{E} = -\mathcal{O} = \triangle + E_\mathcal{E}$ the effective action (performing a Gaussian integration) can be written as the derivative of the zeta function[3] $\zeta_{\mathcal{O}_\mathcal{E}}[s] := \mathrm{tr}\,(\mathcal{O}^{-s})$ for the parameter s:

$$\begin{aligned} W &= \frac{\mathrm{i}}{2}\ln\det(-\mathcal{O}_\mathcal{E}) + \mathrm{const.} \\ &\approx \frac{\mathrm{i}}{2}\mathrm{tr}\ln(-\mathcal{O}_\mathcal{E}) = -\frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}s}\zeta_{\mathcal{O}_\mathcal{E}}[s]\Big|_{s=0} \end{aligned} \tag{4}$$

The zeta function can be expressed by the heat-kernel $\mathrm{e}^{\mathcal{O}_\mathcal{E}\tau}$:

$$\zeta_{\mathcal{O}_\mathcal{E}}[s] = \frac{1}{\Gamma(s)}\int_0^\infty \tau^{s-1}\mathrm{tr}\left(\mathrm{e}^{\mathcal{O}_\mathcal{E}\tau}\right)\mathrm{d}\tau\,. \tag{5}$$

The trace of the heat-kernel in CPTH [11] is expanded in orders of curvature[4]

$$\mathrm{tr}\left(\mathrm{e}^{\mathcal{O}_\mathcal{E}\tau}\right) = \frac{1}{4\pi\tau}\int_\mathcal{E}\left\{a_0 + \tau a_1 + \tau^2 a_2 + \dots\right\}\sqrt{g}\mathrm{d}^2x_\mathcal{E}\,, \tag{6}$$

where the zeroth and first order coefficients are local and agree with those of the Seeley–DeWitt expansion [18], namely

$$\begin{aligned} a_0 &= 1\,, \\ a_1 &= \frac{R_\mathcal{E}}{6} + E_\mathcal{E}\,. \end{aligned} \tag{7}$$

All other coefficients are non-local, i.e. they contain integrations over the Green function on the Euclidean spacetime $L_\mathcal{E}$, expressed by inverse powers of $\Box$ (Green functions) and the presence of form factors like f (cf. [2], (18)):

$$\begin{aligned} a_2 = {} & R_\mathcal{E}\left[\frac{1}{16\tau\Box} + \frac{f(\tau\Box)}{32} + \frac{f(\tau\Box)-1}{8\tau\Box} + \frac{3[f(\tau\Box)-1]}{8(\tau\Box)^2}\right] \\ & \times R_\mathcal{E} + E_\mathcal{E}\left[\frac{f(\tau\Box)}{6} + \frac{f(\tau\Box)-1}{2\tau\Box}\right]R_\mathcal{E} \\ & + R_\mathcal{E}\frac{f(\tau\Box)}{12}E_\mathcal{E} + E_\mathcal{E}\frac{f(\tau\Box)}{2}E_\mathcal{E}\,, \end{aligned} \tag{8}$$

$$f(x) = \int_0^1 \mathrm{e}^{-a(1-a)x}\mathrm{d}a\,. \tag{9}$$

A series expansion of the effective action in orders of curvature thus reads

$$\begin{aligned} W = -\frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}s}\Bigg\{ & \frac{1}{\Gamma(s)}\int_0^\infty \frac{\mathrm{d}\tau}{4\pi\tau^{2-s}} \\ & \times \int_\mathcal{E}\left\{a_0 + \tau a_1 + \tau^2 a_2 + \dots\right\}\sqrt{g}\mathrm{d}^2x_\mathcal{E}\Bigg\}\Bigg|_{s=0}\,. \end{aligned} \tag{10}$$

The key feature of CPTH within the context of 2D dilaton gravity is that the effective action W is computed directly from (10) (well-defined in the Boulware state [2]), and not only certain functional derivatives thereof in the conformal gauge, which are integrated later on [3]. It is the (infinite) τ-integral of (10) that makes the IR divergence explicit which in the previous approach [3, 6] was not evident.

In order to be self-contained we quote the result of [2] for the finite part (up to finite renormalization effects from the UV/IR divergences involving renormalization constants c_i) of W to second order of CPTH (with Minkowski signature):

$$W_{\mathrm{finite}} = \frac{1}{96\pi}\int_\mathcal{M}\left[(R+12E)\frac{1}{\Box}R\right]\sqrt{-g}\mathrm{d}^2x\,, \tag{11}$$

which for SRG, (3), reproduces the correct asymptotic Hawking flux of [3].

[3] For simplicity we shall write very often the Minkowski signature operator $\mathcal{O}$ in connection with the zeta function and the heat-kernel where there should be $-\mathcal{O}_\mathcal{E}$.

[4] In this context "curvature" means the scalar curvature R, the endomorphism E as well as some gauge curvature (which in the present case is absent).

3 Divergences

In this section we recall all divergences in W produced by CPTH in two dimensions, including the UV divergence(s) of the lower order(s). Generally one can say that a divergence at the lower limit of the τ-integration corresponds to a UV divergence, while the upper limit may cause IR divergences. In earlier work [2] we used a cut-off T at large τ to control the IR behavior of the heat-kernel. Here we introduce a mass term instead, the advantage with respect to a cut-off in τ being a clear separation between UV and IR divergences.

3.1 UV divergences

From (10) we observe that the zeroth order of CPTH, proportional to a_0, has a pole of first order at $\tau = 0$. In [12] this UV divergence had been dismissed summarily by dimensional regularization arguments (see (3.17) in [12]). In fact, a constant divergence $\propto 1/\varepsilon$, where $\varepsilon > 0$ is a cut-off at $\tau = 0$, can be interpreted as the infinite contribution of the vacuum energy to the cosmological constant:

$$W_{\mathrm{UV}} = W_0 = \frac{1}{8\pi}\int_{\mathcal{M}} \frac{1}{\varepsilon}\sqrt{-g}\mathrm{d}^2x\,. \tag{12}$$

Its infinite contribution to the Hawking flux in SRG could be renormalized by substracting the flat spacetime value [2].

Apart from the one related to a_0 there are no further UV divergences in two dimensions.

3.2 IR divergences

Starting with the first one all orders in CPTH produce IR divergences. It is obvious that the divergence of the first order is logarithmic and simply proportional to a_1. The higher order coefficients include exponential form factors like e.g. (9), leading to logarithmic divergences as well.

It should be noted that for the particular case of $E = 0$ (providing conformal invariance of the two-dimensional action), the effective action becomes IR finite even in two dimensions [19]!

The effective action shows a well-known ambiguity due to translation invariance of the path integral which can be expressed in the form

$$W = -\frac{\mathrm{i}}{2}\frac{\mathrm{d}}{\mathrm{d}s}\zeta_{\mathcal{O}_{\mathcal{E}}}[s]\Big|_{s=0} - \frac{\mathrm{i}}{2}\zeta_{\mathcal{O}_{\mathcal{E}}}[0]\ln\tilde{\mu}^2. \tag{13}$$

Its origin is the possibility[5] to re-define $\mathcal{O}$ by a multiplication with $\tilde{\mu}^{-2}$. This can be shifted into the scalar field S and simply leads to a multiplication of all positive eigenvalues of the elliptic operator $-\mathcal{O}_{\mathcal{E}}$ by this factor and thus to a contribution to the effective action because $\zeta[s] = \mathrm{tr}(\lambda^{-s}) \to \mathrm{tr}[(\lambda/\tilde{\mu}^2)^{-s}]$ [1]. In the present case we have $\zeta_{\mathcal{O}_{\mathcal{E}}}[0] = \frac{1}{4\pi}\int_{L_{\mathcal{E}}} a_1\sqrt{g}\mathrm{d}^2x_{\mathcal{E}}$ and hence by (7) an ambiguity

$$W_{\mathrm{amb}} = -\frac{1}{96\pi}\int_{\mathcal{M}} \left[(2R+12E)\ln\tilde{\mu}^2\right]\sqrt{-g}\mathrm{d}^2x\,. \tag{14}$$

Therefore, at first sight the IR divergence (17) of the first order CPTH could be renormalized by simply adjusting the constant $\tilde{\mu}^2$. However, the latter could be related to the UV renormalization in higher-dimensional theories (as usual in two dimensions UV and IR divergences may combine). Anyway, an ambiguity of the type (14) is not sufficient to renormalize all orders of CPTH.

4 IR regularization

We regularize the effective action by introducing a mass term in the d'Alembertian: $\mathcal{O} \to \mathcal{O} + m^2$. This mass term can be pulled out from the trace of the heat-kernel

$$\mathrm{tr}\left(\mathrm{e}^{-[\mathcal{O}+m^2]\tau}\right) = \mathrm{e}^{-m^2\tau}\mathrm{tr}\left(\mathrm{e}^{\mathcal{O}_{\mathcal{E}}\tau}\right)\,, \tag{15}$$

and thus regularizes the zeta function (5) at the upper limit of τ.

The leading power in the radius r which determines the asymptotic flux for SRG from (11) from the second order in CPTH has been found to be independent of the IR problem. In order to try to understand this we thus have to go beyond that. In the following we compute the IR divergences up to the third order of CPTH. The final expressions are presented in Minkowski signature spacetime. Thereby we pick up a factor i from the Euclidean volume element $\mathrm{d}^2x_{\mathcal{E}} = \mathrm{i}\cdot\mathrm{d}^2x$ which is multiplied by the factor $-\mathrm{i}$ in (10) resulting in no overall sign change. The Euclidean expressions in the zeta function $\zeta[s]$ like metrics, scalar curvature etc. produce a minus sign when switching to Minkowski signature spacetime.

4.1 First order of CPTH

In the first order term $\propto a_1$ a simple substitution of the integration variable $\tau \to m^2\tau$ is sufficient

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{1}{\Gamma(s)}\int_0^\infty \tau^{s-1}\mathrm{e}^{-m^2\tau}\mathrm{d}\tau\right\} &= \frac{\mathrm{d}}{\mathrm{d}s}\left\{(m^2)^{-s}\right\}\\ &= -(m^2)^{-s}\ln m^2\,,\end{aligned} \tag{16}$$

yielding at $s \to 0$ the expected $\ln m^2$ behavior. Going back to Minkowski signature spacetime the resulting contribution W_1 to the effective action reads

$$W_1 = \frac{1}{96\pi}\int_{\mathcal{M}} \left[(2R+12E)\ln m^2\right]\sqrt{-g}\mathrm{d}^2x\,. \tag{17}$$

Replacing all Euclidean by Minkowski signature variables the curvatures acquire a minus sign by this transition:

[5] We emphasize already here that this possibility disappears in the presence of some external source (cf. the last paragraph of Sect. 5).

$R = -R_{\mathcal{E}}, E = -E_{\mathcal{E}}$. Therefore, the choice $\mu^2 = m^2$ in (14) would be sufficient to formally renormalize the first order divergence to zero. However, this would contribute to the inherent mixing of UV and IR renormalization alluded to already above which we want to avoid.

Of course, $R\sqrt{-g}$ is a total divergence in two dimensions and we therefore may omit its contribution to the effective action.

4.2 Second order of CPTH

The second order (8) of CPTH consists of non-local terms only that are produced by the form factor f, (9). As announced we write $\Box$ instead of $-\triangle$ for simplicity, although we should still treat quantities to be Euclidean at intermediate steps. There are five types of heat-kernel integrals contributing to the second order in (8), the most non-trivial one being (see Appendix A, (A.6), (A.8))

$$\begin{aligned} I_f &= \frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{1}{\Gamma(s)}\int_0^\infty \tau^s f(\tau\Box)\mathrm{e}^{-m^2\tau}\mathrm{d}\tau\right\}\Bigg|_{s=0} \\ &= \frac{1}{m^2}F\left(\frac{\Box}{m^2}\right) \\ &= \frac{z\cdot F(z)}{\Box} = \frac{2}{\Box}\frac{\mathrm{Arcosh}\left(\frac{z}{2}+1\right)}{\sqrt{1+\frac{4}{z}}}\,, \end{aligned} \tag{18}$$

where $z = \Box/m^2$. $F(z)$ is a regular function for all $z \in [0,\infty[$ and possesses a series expansion around $z = 0$. This allows for an expansion of $F(\Box/m^2)$ in terms of a complete set of eigenfunctions[6] $\varphi_\lambda(y)$ of $\Box$ which proves useful when examining the action of (18) on some function, say E as in (8):

$$\begin{aligned} I_f E &= \frac{z\cdot F(z)}{\Box}\int_0^\infty \delta(y-y')E(y')\mathrm{d}y' \\ &= \frac{1}{\Box}\int_0^\infty \mathrm{d}\lambda\left(\frac{\Box}{m^2}\right) cdot F\left(\frac{\Box}{m^2}\right)\varphi_\lambda(y) \\ &\quad\times\int_0^\infty \varphi_\lambda(y')\cdot E(y')\mathrm{d}y' \\ &= \frac{4M^2}{\Box}\int_0^\infty \mathrm{d}\lambda \frac{2\mathrm{Arcosh}\left(\frac{\frac{\lambda^2}{4M^2m^2}+2}{2}\right)}{\sqrt{1+\frac{16M^2m^2}{\lambda^2}}}\varphi_\lambda(y) \\ &\quad\times\int_0^\infty \varphi_\lambda(y')\cdot E(y')\mathrm{d}y' \end{aligned} \tag{19}$$

Here and in the following it is convenient to keep $\Box^{-1}$ outside the λ-integral. This seems natural since all (IR regular) second order terms contain one Green function, making them non-local. Its action on the whole expression should be determined by different means, e.g. by letting it act to the left (cf. (6) with (8)).

In a next step we take the limit $m \to 0$ at fixed, finite λ:

$$\frac{\lambda}{m^2}F\left(\frac{\lambda}{m^2}\right) \overset{m\to 0}{\to} -2\ln\left(\frac{m^2}{\lambda}\right) + O(m^2)\,. \tag{20}$$

Whether this really spearates the IR divergence $\propto \ln m^2$ clearly depends on the behavior of the integrand in (19) at $\lambda \to 0$ for which the dependence on λ of the eigenfunctions is crucial. Another way to check the justification of this separation for all values of λ consists in assuming (20) and to verify that the finite remnant, essentially given by $\Box^{-1}\int_0^\infty \mathrm{d}\lambda \ln\lambda\cdot\varphi_\lambda(y)\int_y \varphi_\lambda(y')E(y')\mathrm{d}y'$, is well-defined. An analysis (Appendix C) of the eigenfunctions and of the double integral in (19) suggests that the latter converges for all values of y and in the case of SRG falls off asymptotically at least as y^{-1}. Therefore, the limit (20) can be justified indeed in (18) even including $\lambda \to 0$, yielding the formal limit:

$$I_f \overset{m\to 0}{\to} -2\frac{\ln\left(\frac{m^2}{\Box}\right)}{\Box} + O(m^2) \tag{21}$$

In [2] we already showed that only the last term in (8), which is of the type $E_{\mathcal{E}}\, I_f\, E_{\mathcal{E}}$, leads to the IR divergence of the second order CPTH. Although I_f also appears in other terms in (8) which are of the type $R^2_{\mathcal{E}}$, $R_{\mathcal{E}}\cdot E_{\mathcal{E}}$ those contributions cancel. In comparison with the cut-off regularization $\tau \le T$ of [2] the result (21) is shifted by some finite constant $\ln m^2 = -\ln T - \gamma_{\mathrm{E}}$ where $\gamma_{\mathrm{E}} \approx 0.57721$ is the Euler constant. To check whether the regular part of the second order CPTH remains unchanged, also the integral (see Appendix A, (A.9))

$$\begin{aligned} I_{f-1} &= \frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{1}{\Gamma(s)}\right. \\ &\quad\times\left.\int_0^\infty \tau^{s-1}\frac{\int_0^1 \mathrm{e}^{-a(1-a)\Box\tau}\mathrm{d}a - 1}{\Box}\mathrm{e}^{-m^2\tau}\mathrm{d}\tau\right\}\Bigg|_{s=0} \\ &= -\frac{z\cdot G(z)}{\Box} \end{aligned} \tag{22}$$

is needed. Again the formal limit $m \to 0$ is justified:

$$I_{f-1} \overset{m\to 0}{\to} \frac{\ln\left(\frac{m^2}{\Box}\right)+2}{\Box} + O(m^2) \tag{23}$$

Comparing with τ regularization in [2], (22) is shifted by the same amount as in (21), and therefore (one piece[7] of) the regular part is in perfect agreement with [2] the terms $\propto E_{\mathcal{E}}\cdot R_{\mathcal{E}}$ in (8):

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{1}{\Gamma(s)}\int_0^\infty \tau^{s-1}\mathrm{d}\tau\right\} E_{\mathcal{E}}\left[\frac{f(\tau\Box)}{4} + \frac{f(\tau\Box)-1}{2\tau\Box}\right] R_{\mathcal{E}} \\ &\quad = E_{\mathcal{E}}\frac{1}{\Box}R_{\mathcal{E}}\,. \end{aligned} \tag{24}$$

[6] For SRG the dimensionless variable $y = \frac{r}{2M} - 1$ is related to the radius measured from the horizon $2M$, $\lambda \ge 0$ is a properly defined dimensionless eigenvalue (cf. Appendix C).

[7] There is another regular non-local contribution $\propto R^2_{\mathcal{E}}$ to this order of CPTH [12] (the first line in (8)) which is not considered here.

This proves the concordance of the mass regularization with the cut-off one in the regular sector, yielding the IR divergent contribution

$$W_2 = \frac{1}{96\pi}\int_{\mathcal{M}}\left[-12\cdot E\frac{\ln\left(\frac{m^2}{\Box}\right)}{\Box}E\right]\sqrt{-g}\mathrm{d}^2x \qquad (25)$$

to the effective action.

4.3 Third order of CPTH and summation

To third order of CPTH technical complications increase dramatically. Therefore, we make the assumption that the IR divergence is contained in the "pure" endomorphism term which is $\propto E^3$. This is supported by the fact that

– the IR divergences were "purely endomorphism" to the first two orders;
– the effective action becomes IR finite in the case $E = 0$ [19].

In Appendix B we show that (under this assumption) the unique, logarithmically divergent part of the endomorphism term is given by

$$W_3 = \frac{1}{96\pi}\int_{\mathcal{M}}\left[12\cdot\ln m^2\cdot E\frac{1}{\Box}E\frac{1}{\Box}E\right]\sqrt{-g}\mathrm{d}^2x\,. \qquad (26)$$

Here we have disregarded terms formally written as $\ln\Box$ as we are mainly interested in the IR structure. The proper interpretation of such terms, being IR regular, is discussed in Appendix C.

Inspection of the IR divergent terms (17), (25), and (26) reveals a nice pattern, suggesting a similar structure for the higher order ones in an infinite series, which even lends itself to a formal summation. It proves useful to introduce some renormalization parameter μ^2 by replacing $\ln m^2 = \ln(m^2/\mu^2) - \ln\mu^2$ and $\ln(m^2/\Box) = \ln(m^2/\mu^2) - \ln(\mu^2/\Box)$, respectively. Adding (17), (25), and (26) the IR divergence of the effective action (up to the third order of CPTH) can be given in the form

$$W_{\mathrm{IR}} = \frac{1}{8\pi}\int_{\mathcal{M}}\left[\left(E - E\frac{1}{\Box}E + E\frac{1}{\Box}E\frac{1}{\Box}E\right)\ln\frac{m^2}{\mu^2}\right] \times\sqrt{-g}\mathrm{d}^2x\,. \qquad (27)$$

It is striking that these terms can be reproduced by the formal series expansion

$$\Box\frac{1}{\mathcal{O}}E = \Box\frac{1}{\Box+E}E = \Box\frac{1}{1+\frac{1}{\Box}E}\frac{1}{\Box}E = \Box\left(1 - \frac{1}{\Box}E + \frac{1}{\Box}E\frac{1}{\Box}E + \dots\right)\frac{1}{\Box}E = E - E\frac{1}{\Box}E + E\frac{1}{\Box}E\frac{1}{\Box}E + \dots\,. \qquad (28)$$

This suggests that the total IR divergence of the effective action can be represented formally as

$$W_{\mathrm{IR}} = -\frac{1}{8\pi}\int_{\mathcal{M}}\left\{\Box\frac{1}{\mathcal{O}}E\right\}\cdot\ln\xi\,\sqrt{-g}\mathrm{d}^2x\,, \qquad (29)$$

where $\xi = \mu^2/m^2 \to +\infty$ in the limit $m^2 \to 0$. Expression (29) indeed produces divergent contributions to expectation values as for instance the "dilaton anomaly" in SRG [3]:

$$\left\langle T^\theta{}_\theta\right\rangle_2 \propto \frac{\delta W_{\mathrm{IR}}}{\delta\phi} = \int_{\mathcal{M}}\frac{\delta W_{\mathrm{IR}}}{\delta E}\frac{\delta E}{\delta\phi}\sqrt{-g}\mathrm{d}^2x = \frac{1}{8\pi}\int_{\mathcal{M}}\frac{\delta E}{\delta\phi}\frac{\ln\xi}{(1+\rho)^2}\sqrt{-g}\mathrm{d}^2x\,. \qquad (30)$$

Here we have used the SRG conformal gauge[8] representation $E = 2M/r^3 = \Box\rho$ (implying $\phi = -\ln r$ in (3) [2]. However, as it should be, the IR divergence (29) does not contribute to the trace anomaly[9].

Of course, a mathematically stringent discussion of the convergence for the series expansion (28) is impossible. Nevertheless, we add here some heuristic argument. A necessary formal condition is $|\Box^{-1}E| < 1$. In the case of SRG we have $\Box^{-1}E = \rho = \ln\left(1 - \frac{2M}{r}\right)/2$, and therefore this condition is fulfilled for $r > 1.156\cdot r_{\mathrm{h}}$ where $r_{\mathrm{h}} = 2M$ is the radius of the event horizon. Thus the series may converge a short distance outside the horizon, excluding thereby, however, the most interesting region. On the other hand, it could be speculated that the logarithmic divergence at the horizon [6–8] when the leading flux term is extrapolated back may be related to (or even compensated by) a divergence of the series (28).

5 Renormalization

We have shown that the formal expression (29) correctly reproduces the IR divergences of the CPTH up to the second order and at least partly to the third order. The next task is whether one can find physically reasonable counterterms in order to eliminate (29).

To this account we introduce a source term jS beside the scalar action in the path integral (1):

$$L[g,S] = \int_{\mathcal{M}}\left[jS - \frac{1}{2}S\mathcal{O}S\right]\sqrt{-g}\mathrm{d}^2x := -\frac{1}{2}\int_{\mathcal{M}}\left[\hat{S}\mathcal{O}\hat{S} + j\mathcal{O}^{-1}j\right]\sqrt{-g}\mathrm{d}^2x\,. \qquad (31)$$

The current j has been shifted into the re-defined scalar field $\hat{S}$ by translation invariance of the path integral. Now, by rewriting the inverse operator as $\mathcal{O}^{-1} = \Box^{-1}\Box\mathcal{O}^{-1}\Box\Box^{-1}$, where $\Box\mathcal{O}^{-1}\Box$ can be expanded

[8] Conformal gauge is defined by $g_{\alpha\beta} = \mathrm{e}^{2\rho}\eta_{\alpha\beta}$ generally and $\rho = \ln\left(1 - \frac{2M}{r}\right)/2$ in the particular case of SRG. The implicit dependance on the tortoise coordinate $r(r_*)$ is not relevant here.

[9] In this context note that the representation $E = \Box\rho$ of the endomorphism can be used only after variation of the effective action (before that we must use (3)). Therefore, (29) does not contain the conformal factor ρ explicitly and hence is conformally invariant.

in a series similar to (28), one obtains for the source term W_j of the effective action

$$
\begin{aligned}
W_j[g,j] &= -\frac{1}{2}\int_{\mathcal{M}} j\frac{1}{\Box}\left(\Box\frac{1}{\mathcal{O}}\Box\right)\frac{1}{\Box}j\sqrt{-g}\mathrm{d}^2x \\
&= -\frac{1}{2}\int_x\int_y\int_z j(x)G(x,y) \\
&\quad\times\left(\Box - E + E\frac{1}{\Box}E - \dots\right)_y G(y,z)j(z) \\
&= -\frac{1}{2}\int_{\mathcal{M}} j\frac{1}{\Box}j\sqrt{-g}\mathrm{d}^2x \\
&\quad+\frac{1}{2}\int_x\int_y\int_z\left(E - E\frac{1}{\Box}E + \dots\right)_y \\
&\quad\times G(y,x)j(x)G(y,z)j(z) \\
&= -\frac{1}{2}\int_{\mathcal{M}} j\frac{1}{\Box}j\sqrt{-g}\mathrm{d}^2x \\
&\quad+\frac{1}{2}\int_{\mathcal{M}}\left(E - E\frac{1}{\Box}E + \dots\right)\left(\frac{1}{\Box}j\right)\left(\frac{1}{\Box}j\right) \\
&\quad\times\sqrt{-g}\mathrm{d}^2y\,.
\end{aligned}
\tag{32}
$$

For clarity in the intermediate step of (32) we wrote the Green function $G(x,y)$, as defined by $\Box G(x,y) = -\delta^2(x-y)$, instead of $-\Box^{-1}$. We furthermore assumed it to be symmetric in its arguments[10]. We now re-define the source j by adding a term $-\Box\chi_0$. Then, up to terms that vanish in the limit $j \to 0$, a contribution quadratic in χ_0 survives:

$$
\begin{aligned}
W_j[g,j] &= \frac{1}{2}\int_{\mathcal{M}}\left[\left(E - E\frac{1}{\Box}E + \dots\right)\chi_0^2 - \chi_0\Box\chi_0\right] \\
&\quad\times\sqrt{-g}\mathrm{d}^2x + O(j)\,,
\end{aligned}
\tag{33}
$$

where the terms in the brackets are precisely of the form (28) of the IR divergence (29). Thus the latter can be removed to all orders of CPTH by the choice

$$
\chi_0^2 := \frac{\ln\xi}{4\pi} = \frac{\ln\left(\frac{\mu^2}{m^2}\right)}{4\pi} > 0\,. \tag{34}
$$

We emphasize that even the sign is consistent as χ_0^2 clearly must be positive. For such a zero-mode χ_0 the additional term $\propto \chi_0\Box\chi_0$ in (33) vanishes.

As for constant χ_0 we have trivially $\Box\chi_0 = 0$, at first sight our procedure seems to be very strange. Indeed, reconsidering the classical action (31) with source term jS before redefining the scalar field S, a shift in the source $j \to j - \Box\chi_0$ leaves it invariant if χ_0 is a constant (or a zero-mode). However, this shift affects the effective action containing combinations $\Box^{-1}j$, i.e. a non-local effect occurs. Therefore, the contribution (33) to the effective action must be considered some quantum ambiguity similar to the well-known one described in (13) and (14) above. There we saw that the latter could be used to renormalize (17), the first order of CPTH, only. In this respect the present ambiguity appears to be an extension of that to all orders of CPTH, whereby the origin again could be found in the translation invariance of the path integral. It is, however, powerful enough to remove all IR divergences of the theory (assuming that our conjecture upon the higher order terms is valid). Thereby it introduces the renormalization constant μ^2 into the remaining finite part of the effective action which should be determined by the value of the Hawking flux.

A crucial difference between the former (14) and the new ambiguity (33) is the relation to a source term in the case of the latter. In the presence of this source term the former "symmetry" of the path integral under multiplication of $\mathcal{O}$ by some renormalization constant, which led to the original ambiguity (14), is destroyed. Therefore it seems that one has to choose between two kinds of ambiguities, whereby only the one exhibited here allows for a complete IR renormalization to all orders of CPTH which works by factorising out the IR terms.

[10] This restricts our argument to the Feynman Green function which, anyway, is naturally preferred due to the Euclidean analysis in the derivation of the effective action. Here we differ from Barvinsky and Vilkovisky who argue that the retarded Green function should be inserted "by hand" [11]. In the context of SRG this step could not be confirmed [2].

6 Conclusions

The aim of this paper was to shed new light on the infrared problem inherent in general scalar effective actions in two dimensions. To establish the effective action we used the covariant perturbation theory of Barvinsky and Vilkovisky [11], based upon an expansion in orders of the curvature. Curvature in this context not only refers to the scalar curvature R associated with the d'Alembertian $\Box$ but also to some potential E called endomorphism "acting" on the scalar field S and thus forming a general d'Alembertian $\mathcal{O} = \Box + E$.

We further added a mass term to control the infrared divergences of the effective action (15). The calculation of these divergences has been performed in detail for the second and, with simplifying assumptions, also to third order of covariant perturbation theory. In comparison with a previous approach [2] (where a cut-off of the eigentime τ in the heat-kernel formalism was used as a regulator) the results for the integrals were the same up to a finite constant. The latter is irrelevant as it does not enter the regular part of the effective action (24). The key problem has been the separation of the infrared divergences proportional to $\ln m^2$ which, starting with the second order of covariant perturbation theory, are coupled to formal terms $\ln\Box$. Going back to the origin of the latter and using an expansion in terms of eigenfunctions of (the radial part of) the d'Alembertian we could show that the separation of the IR divergences produced finite remnants of the same type, including some renormalization constant μ^2.

The infrared divergences of the first three orders of covariant perturbation theory were shown to fit into the pattern of a formal series (28) that could be summed into a single term (29), containing the inverse of the general d'Alembertian $\mathcal{O}$. At this point we conjectured the extension of this identity to higher orders.

The introduction of a source term jS (31) in the path integral revealed the existence of an ambiguity (33) of the effective action due to its non-locality. A shift in the source $j \to j - \Box\chi_0$ leaves the classical action invariant but produces non-vanishing contributions to the effective action if χ_0 is a zero-mode of the d'Alembertian. The resulting contribution exhibits the same structure as the infrared divergence (29) but carries the opposite sign and thus proves a good candidate for renormalization. Indeed, by a choice $\chi_0^2 = \ln\xi/4\pi$ where $\xi = \mu^2/m^2 > 0$, the total infrared divergence of the effective action can be renormalized, thereby leaving finite terms containing some constant μ^2. The latter appear in new types of terms in the effective action which had not been considered previously ($\propto \ln\Box/\mu^2$), and should be fixed by physical observables like the Hawking flux in the case of spherically reduced gravity.

A particular feature of these renormalization terms is that they are not conformally invariant and thus lead to contributions to the trace anomaly. A heuristic argument suggests that they contribute significantly only in the region where the convergence of the series (28) breaks down (i.e. close to the horizon). There they may even lead to a logarithmic divergence, similar to the one found in the "dilaton anomaly" [3]. This behavior might hint at an analogous problem of convergence in the regular sector of the covariant perturbation theory, which, however, is more difficult to investigate since the coefficients show no comparable, simple pattern. On the other hand it might even be that the logarithmic divergences produced by the renormalization terms compensate the one of the dilaton anomaly.

For vanishing endomorphism E the infrared divergence (29) becomes a trivial surface term and hence vanishes. This is in agreement with the observation that the effective action should be finite in two dimensions in this particular case [19].

It is a peculiarity of our approach that the shift in the source j produces a term linear in j and thus a non-vanishing expectation value $\langle S\rangle$. By construction it is a solution of the d'Alembertian $\mathcal{O}$ because $\mathcal{O}\langle S\rangle = \Box\chi_0 = 0$ and approaches asymptotically the zero-mode $\langle S\rangle \overset{r\to\infty}{\to} \chi_0$.

The present calculations clearly cannot prove the assumptions made about the summation of the IR divergences to all orders as they are based upon arguments only involving terms up to the third order. However, the cancellation of divergences up to the second and (at least partly) to the third order by means of our approach is manifest and thus strongly supports the results on the Hawking flux obtained by this method [2].

We found that the IR divergent terms, as well as their compensation, can be collected in a factor. Such factorizations of IR terms in the limit of small masses are a phenomenon which is known for a long time in connection with the emission of soft photons [20, 21]. It is amusing that a similar phenomenon seems to occur here, and this may suggest an analogous renormalization prescription.

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Appendix A: Heat-kernel integrals to the second order of CPTH – mass term regularization

The exponential function in $f(\tau\Box)$ in (18) (which in the following we call I_f) can be expanded formally in a series:

$$I_f = \sum_{n=0}^{\infty}(-1)^n \frac{c_n}{n!}\Box^n \frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{1}{\Gamma(s)}\int_0^\infty \tau^{s+n}\mathrm{e}^{-m^2\tau}\mathrm{d}\tau\right\}\Bigg|_{s=0}, \tag{A.1}$$

where $c_n = \int_0^1[a(1-a)]^n\mathrm{d}a = (1/4)^n\int_0^1(1-u^2)^n\mathrm{d}u$. In the integral

$$C_n(k) := \int_0^k (k^2-u^2)^n\mathrm{d}u = k^{2n+1}C_n(1) \tag{A.2}$$

differentiation by k gives

$$\frac{\mathrm{d}C_n(k)}{\mathrm{d}k} = \int_0^k n(k^2-u^2)^{n-1}2k\,\mathrm{d}u, = (2n+1)k^{2n}C_n(1)\,, \tag{A.3}$$

and for $k=1$ we have the recursion formula

$$(2n+1)C_n(1) = 2n\cdot C_{n-1}(1)\,. \tag{A.4}$$

Since $c_n = (1/4)^nC_n(1)$ this becomes a similar relation for the c_n:

$$c_{n+1} = \frac{n+1}{4n+6}\cdot c_n \tag{A.5}$$

Using $\int_0^\infty \mathrm{e}^{-m^2\tau}\tau^{s+n}\mathrm{d}\tau = \frac{\Gamma(s+n+1)}{m^{2(s+n+1)}}$ we can perform the τ-integration of I_f:

$$I_f = \frac{1}{m^2}\sum_{n=0}^{\infty}(-1)^n c_n z^n := \frac{z\cdot F(z)}{\Box}\,, \tag{A.6}$$

where we have defined the function $F(z)$ and $z := \Box/m^2$. In order to obtain a closed expression for F the recursion relation (A.5) can be used to establish a differential equation

$$F'(4z+z^2) + F(2+z) = 2 \tag{A.7}$$

with general solution

$$F(z) = \frac{2\mathrm{Arcosh}\left(\frac{z+2}{2}\right)}{\sqrt{4z+z^2}} + \frac{c}{\sqrt{4z+z^2}} \tag{A.8}$$

depending on an integration constant c. I_f vanishes for $m\to\infty$ or $z\to 0$, which entails $c=0$, producing (18).

Performing exactly the same steps as before we can calculate the heat-kernel integral I_{f-1} of (22) with the function

$$G(z) = \sum_{n=0}^{\infty} (-1)^n b_n z^n \,, \; b_n = \frac{n}{2(2n+3)} b_{n-1} \,. \qquad \text{(A.9)}$$

Appendix B: Third order of CPTH

To calculate the IR divergent term of order E^3 in CPTH we start from (see (2.28) in [11])[11]

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left\{ \frac{1}{\Gamma(s)} \int_0^\infty \tau^{s-1} \mathrm{d}\tau \frac{\tau^3}{3\tau^1} \int_0^1 \mathrm{d}\alpha \int_0^\alpha \mathrm{d}\beta \right.\\
&\times \int_{L_\mathcal{E}} \sqrt{g}\mathrm{d}^2 x \mathrm{e}^{-\tau\left[\beta(\alpha-\beta)\Box_1 + \beta(1-\alpha)\Box_2 + (\alpha-\beta)(1-\alpha)\Box_3 + m^2\right]}\\
&\left.\times E_1 E_2 E_3 \right\}\Big|_{s=0}\\
&= \frac{\mathrm{d}}{\mathrm{d}s}\left\{ \int_\alpha \int_\beta \int_{L_\mathcal{E}} \left[\beta(\alpha-\beta)\Box_1 + \beta(1-\alpha)\Box_2 \right.\right. .\\
&\quad \left. + (\alpha-\beta)(1-\alpha)\Box_3 + m^2\right]^{-s-2}\\
&\left.\times \cdot \frac{E_1 E_2 E_3}{3} \cdot s(s+1) \right\}\Big|_{s=0} \,. \qquad \text{(B.1)}
\end{aligned}$$

The β-integration is done first, using the formula

$$\int \frac{1}{X^n} \mathrm{d}x = \frac{2ax+b}{(n-1)\Delta X^{n-1}} + \frac{(2n-3)2a}{(n-1)\Delta} \int \frac{\mathrm{d}x}{X^{n-1}} \,, \qquad \text{(B.2)}$$

where we have $a = -\Box_1, b = \alpha\Box_1 + (1-\alpha)[\Box_2 - \Box_3], c = \alpha(1-\alpha)\Box_3 + m^2$ and $\Delta = -\alpha^2\Box_1^2 - (1-\alpha)^2[\Box_2 - \Box_3]^2 - 2\alpha(1-\alpha)\Box_1[\Box_2 - \Box_3] - 4m^2\Box_1$. m^2 can be set to zero in Δ, because it leaves the whole expression regular. This can be checked by inserting $\alpha = 0, 1$. The β-integration yields

$$\begin{aligned}
\int_0^\alpha Y^{-s-2} \mathrm{d}\beta &= \frac{-2\Box_1\beta + \alpha\Box_1 + (1-\alpha)[\Box_2 - \Box_3]}{(s+1)\Delta \cdot Y^{s+1}}\Big|_0^\alpha\\
&\quad - \frac{(2s+1)2\Box_1}{(s+1)\Delta} \int_0^\alpha Y^{-s-1} \mathrm{d}\beta \,, \qquad \text{(B.3)}
\end{aligned}$$

where $Y = \beta(\alpha-\beta)\Box_1 + \beta(1-\alpha)\Box_2 + (\alpha-\beta)(1-\alpha)\Box_3 + m^2$. Only the surface term retains the IR divergence while the remaining integral will become regular after the α-integration. The upper limit contribution $\beta = \alpha$ is

$$\frac{-\alpha\Box_1 + (1-\alpha)[\Box_2 - \Box_3]}{(s+1)\Delta[\alpha(1-\alpha)\Box_2 + m^2]^{s+1}} \,, \qquad \text{(B.4)}$$

while the lower limit $\beta = 0$ gives

$$-\frac{\alpha\Box_1 + (1-\alpha)[\Box_2 - \Box_3]}{(s+1)\Delta[\alpha(1-\alpha)\Box_3 + m^2]^{s+1}} \,, \qquad \text{(B.5)}$$

these contributions being symmetric under the exchange $2 \leftrightarrow 3$. Next we perform the α-integration for the upper limit contribution, partially integrating the IR divergent expression $[\alpha(1-\alpha)\Box_2 + m^2]^{-s-1}$. The remaining term, being differentiated, cannot lead to further IR divergences because Δ^{-1} is finite on the whole α-interval. For the partial α-integration we have $a = -\Box_2, b = \Box_2, c = m^2$, $\Delta_\alpha = -\Box_2^2 + O(m^2)$ and $X = \alpha(1-\alpha)\Box_2 + m^2$. We have

$$\begin{aligned}
&\int_0^1 \frac{-\alpha\Box_1 + (1-\alpha)[\Box_2 - \Box_3]}{(s+1)\Delta} \left[\alpha(1-\alpha)\Box_2 + m^2\right]^{-s-1} \mathrm{d}\alpha\\
&= \left\{ \frac{-\alpha\Box_1 + (1-\alpha)[\Box_2 - \Box_3]}{(s+1)\Delta} \right.\\
&\quad \left. \times \left[\frac{-2\Box_2\alpha + \Box_2}{-s\Box_2^2} X^{-s} + \frac{-(2s-1)2\Box_2}{-s\Box_2^2} \int X^{-s} \mathrm{d}\alpha \right] \right\}_0^1\\
&\quad - \int_0^1 \left[\frac{-2\Box_2\alpha + \Box_2}{-s\Box_2^2} X^{-s} + \frac{-(2s-1)2\Box_2}{-s\Box_2^2} \int X^{-s} \mathrm{d}\alpha \right]\\
&\quad \times \partial_\alpha \frac{-\alpha\Box_1 + (1-\alpha)[\Box_2 - \Box_3]}{(s+1)\Delta} \mathrm{d}\alpha \qquad \text{(B.6)}
\end{aligned}$$

All terms contain a factor $[s(s+1)]^{-1}$ which is cancelled by an identical one in (B.1). The differentiation for s thus only acts on factors $2s-1$ in the numerators, leaving such expressions harmless, and on X^{-s}, thereby producing a logarithm of the argument. If the latter is evaluated directly on the boundary we arrive at a logarithmic divergence. The remaining integral is regular. Hence, the logarithmic divergence only appears in the first term of the second line in (B.6). We first evaluate it at the boundary:

$$\begin{aligned}
&\left(\frac{-\alpha\Box_1 + (1-\alpha)[\Box_2 - \Box_3]}{(s+1)\Delta} \right) \left(\frac{-2\Box_2\alpha + \Box_2}{-s\Box_2^2} \right)\\
&\quad \times \left[\alpha(1-\alpha)\Box_2 + m^2\right]^{-s} \Big|_0^1\\
&= \left[\frac{1}{\Box_1\Box_2} + \frac{1}{(\Box_3 - \Box_2)\Box_2} \right] \frac{(m^2)^{-s}}{s(s+1)} \qquad \text{(B.7)}
\end{aligned}$$

By the symmetry $2 \leftrightarrow 3$ the lower limit of the β-integration yields an IR divergent term

$$\left[\frac{1}{\Box_1\Box_3} + \frac{1}{(\Box_2 - \Box_3)\Box_3} \right] \frac{(m^2)^{-s}}{s(s+1)} \qquad \text{(B.8)}$$

after the α-integration has been performed. Finally we put these terms into (B.1) and obtain the unique IR divergent contribution of the order E^3:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left\{ (m^2)^{-s} \int_{L_\mathcal{E}} \sqrt{g}\mathrm{d}^2 x \left(\frac{1}{\Box_1\Box_2} + \frac{1}{\Box_1\Box_3} + \frac{1}{\Box_2\Box_3} \right) \right.\\
&\quad \left. \times \frac{E_1 E_2 E_3}{3} \Big|_{\{1,2,3\}=1} \right\}\\
&\overset{s\to 0}{\to} -\ln m^2 \int_{L_\mathcal{E}} E \frac{1}{\Box} E \frac{1}{\Box} E \sqrt{g} \mathrm{d}^2 x \,. \qquad \text{(B.9)}
\end{aligned}$$

[11] A d'Alembertian with index is supposed to act only on functions with the same index, i.e. $\Box_1 E_1 := \lim_{x_1 \to x} \Box_{x_1} E(x_1)$.

Appendix C: Regularity of $\ln\Box$

The regularity of the formal expression $\ln\Box$ in (21) must be discussed for the combination $(\ln\Box/\Box)E$ which appears in the CPTH to second and higher order. An explicit expression can be found by inserting eigenfunctions $\varphi_\lambda(y)$ of the d'Alembertian. Since the action of $\ln\Box$ on the eigenfunctions is not yet defined, we must go back to the function $F(z)$ (see (A.6), Appendix A) which produces the term $\propto(\ln\Box/\Box)E$ in the limit $m\to 0$. Expanding $z\cdot F(z)$ in a power series in $z=\Box/m^2$ its action on the eigenfunctions is well-defined and yields (19).

Eigenfunctions and eigenvalues are defined as $\Box\varphi_\lambda(y)=\tilde{\lambda}^2\ \varphi_\lambda(y)=\lambda^2/(4M^2)\ \varphi_\lambda(y)$ where λ is dimensionless (see below). We have assumed that E is time-independent as it is the case for SRG. In a next step we would like to perform the limit $m^2\to 0$ in the integrand of (19) to isolate the IR divergence $\propto\ln m^2$. Such a result, however, critically depends on the regularity of the λ-integration, in particular on the behavior of the eigenfunctions $\varphi_\lambda(y)$ in the limit $\lambda\to 0$.

C.1 Eigenfunctions

The eigenvalue equation of the radial d'Alembertian

$$\Box_r\varphi_\lambda(r)=-\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(1-\frac{2M}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r}\right]\varphi_\lambda(r)=\tilde{\lambda}^2\cdot\varphi_\lambda(r) \tag{C.1}$$

with the dimensionless radius variable $y:=\frac{r}{2M}-1$ can be brought to the dimensionless form

$$-\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{y}{1+y}\frac{\mathrm{d}}{\mathrm{d}y}\right)\varphi_\lambda(y)=\lambda^2\cdot\varphi_\lambda(y) \tag{C.2}$$

with the dimensionless eigenvalue $\lambda^2:=4M^2\tilde{\lambda}^2$. The differential equation (C.2) possesses two inessential singularities at $y=-1$ and $y=0$, but an essential one at $y\to\infty$. Its solutions, therefore, do not belong to Fuchs' class. At the horizon ($y=0$) (C.2) has two independent solutions, one of which is logarithmically divergent for $y\to 0$. They can be determined by standard methods as generalized power series:

$$\varphi_\lambda^{(1)}=\ 1-\lambda^2y+\frac{\lambda^2(\lambda^2-2)}{4}y^2-\frac{\lambda^4(\lambda^2-8)}{36}y^3+O\left(y^4\right)\,, \tag{C.3}$$

$$\varphi_\lambda^{(2)}=\ \varphi_\lambda^{(1)}\cdot\ln y+(2\lambda^2+2)y-\frac{\lambda^2(3\lambda^2-2)}{4}y^2+O\left(y^3\right)\,. \tag{C.4}$$

Expanding the whole equation (rather than the solution) for small values of y the approximative equation $\varphi'_\lambda+\lambda^2\varphi_\lambda=0$ has the unique solution

$$\varphi_\lambda^h(y)\overset{y\to 0}{\approx}c(\lambda)\cdot\mathrm{e}^{-\lambda^2y}=c(\lambda)\cdot\mathrm{e}^{-\lambda^2(\frac{r}{2M}-1)}\,, \tag{C.5}$$

which is regular at the horizon and approaches there the regular one $\varphi_\lambda^{(1)}$ of the two solutions of the exact eigenvalue- equation (C.2). $c(\lambda)$ is an unknown normalization factor. For large y the solutions

$$\varphi_\lambda^\infty(y)\overset{y\to\infty}{\approx}a(\lambda)\sin(\lambda y)+b(\lambda)\cos(\lambda y) \tag{C.6}$$

correspond to the expected free wave at asymptotic distances. In principle the normalization factors $c(\lambda)$, $a(\lambda)$, and $b(\lambda)$ can be fixed by the orthonormality condition

$$\int_0^\infty\varphi_{\lambda'}\varphi_\lambda\mathrm{d}y=\frac{1}{\Delta\lambda^2}\frac{y}{1+y}\left[\varphi_{\lambda'}\left(\varphi'_\lambda\right)-\left(\varphi'_{\lambda'}\right)\varphi_\lambda\right]\Big|_0^\infty$$
$$:=\delta(\Delta\lambda)\,, \tag{C.7}$$

following from (C.2), where $\Delta\lambda=\lambda'-\lambda$ and $\Delta\lambda^2=(\lambda')^2-\lambda^2$. Because of the pre-factor $y/(1+y)$ only the upper boundary contributes where $\varphi_\lambda^{(1)}$, the regular solution, behaves as (C.6). It is convenient to compare (C.7) to the orthogonality condition of the exactly known eigenfunctions $\varphi_\lambda^0=a(\lambda)\sin(\lambda y)+b(\lambda)\cos(\lambda y)$ of the flat radial d'Alembertian $\Box_0=-\frac{\mathrm{d}^2}{\mathrm{d}y^2}$:

$$\int_0^\infty\varphi_{\lambda'}^0\varphi_\lambda^0\mathrm{d}y=\frac{1}{\Delta\lambda^2}\left[\varphi_{\lambda'}^0\left(\frac{\mathrm{d}}{\mathrm{d}y}\varphi_\lambda^0\right)-\left(\frac{\mathrm{d}}{\mathrm{d}y}\varphi_{\lambda'}^0\right)\varphi_\lambda^0\right]\Big|_0^\infty$$
$$=\frac{\pi}{2}[a^2(\lambda)+b^2(\lambda)]\delta(\Delta\lambda)+\frac{\lambda'a(\lambda')b(\lambda)-\lambda a(\lambda)b(\lambda')}{\Delta\lambda^2}\,. \tag{C.8}$$

The appearance of the last term in (C.8) expresses the fact that the flat eigenfunctions φ_λ^0 are not orthogonal on the half-line. More precisely, the $\sin(\lambda y)$-modes are not orthogonal to the $\cos(\lambda y)$-modes[12].

The upper limit contributions of (C.7) and (C.8) are identical. The lower limit $y=0$ in (C.6) vanishes for φ_λ. Thus we can write

$$\int_0^\infty\varphi_{\lambda'}\varphi_\lambda\mathrm{d}y=\int_0^\infty\varphi_{\lambda'}^0\varphi_\lambda^0\mathrm{d}y+\frac{1}{\Delta\lambda^2}$$
$$\times\left[\varphi_{\lambda'}^0\left(\partial_y\varphi_\lambda^0\right)-\left(\partial_y\varphi_{\lambda'}^0\right)\varphi_\lambda^0\right]\Big|^0$$
$$=\frac{\pi}{2}[a^2(\lambda)+b^2(\lambda)]\delta(\Delta\lambda)\,. \tag{C.9}$$

The lower limit contribution (which can be calculated directly inserting the flat eigenfunctions φ_λ^0) just happens to cancel the inconvenient last term in (C.8). Therefore, the exact eigenfunctions φ_λ of the curved d'Alembertian are orthogonal on the half-line[13] $y\in[0,\infty[$. According to (C.9) the normalization is fixed by a choice

[12] For instance, the term $a(\lambda')b(\lambda)\int_0^\infty\sin(\lambda'y)\sin(\lambda y)\mathrm{d}y$ has been calculated by differentiating the identity $\int_0^\infty\sin y\cos(a\,y)/y\mathrm{d}y=\pi/4[1-\mathrm{sgn}(a-1)]$ for a. For the term $a(\lambda')b(\lambda)\int_0^\infty\sin(\lambda'y)\cos(\lambda y)\mathrm{d}y$, spoiling the orthogonality, we differentiate $\int_0^\infty[\cos(ay)-\cos(by)]/y\mathrm{d}y=\ln b/a$ by a and set $a=\lambda-\lambda',b=\lambda+\lambda'$.

[13] A general Sturm–Liouville operator of the type $-\frac{\mathrm{d}}{\mathrm{d}y}(g(y)\frac{\mathrm{d}}{\mathrm{d}y})$ shares this property, if the function $g(y)$ satisfies $g(\infty)=1$ and $g(0)=0$ as in our case (C.2).

$$a(\lambda) = \sqrt{\frac{2}{\pi}} \cdot \hat{a}\ ,\ \ b(\lambda) = \sqrt{\frac{2}{\pi}} \cdot \hat{b} \qquad \text{(C.10)}$$

for the factors of the large y solutions (C.6), where $\hat{a}^2 + \hat{b}^2 = 1$. The normalization $c(\lambda)$ of the unique exponential solution (C.5) at small values of y, however, is still unknown. As the differential equation (C.2) does not belong to Fuchs' class, the relation between solutions at different singularities is not known. Instead we try to find an answer by an approximate patching of the solutions at some intermediate value $\bar{y}$. Considering (C.2) it is evident that the solutions $\varphi^h_\lambda, \varphi^\infty_\lambda$ are accurate for $y \ll 1$, $y \gg 1$, respectively. Therefore, the choice $\bar{y} = 1$ seems appropriate for the patching. We assume that at this point the regular solution (C.5) and its derivative with $c(\lambda) = \sqrt{2/\pi}\,\hat{c}$ shall approach a linear combination of the two large y solutions:

$$\hat{c} \cdot \mathrm{e}^{-\lambda^2} \approx \left[\hat{a}\sin(\lambda) + \hat{b}\cos(\lambda)\right] \qquad \text{(C.11)}$$

$$-\lambda^2\,\hat{c} \cdot \mathrm{e}^{-\lambda^2} \approx \lambda\left[\hat{a}\cos(\lambda) - \hat{b}\sin(\lambda)\right] \qquad \text{(C.12)}$$

The normalizations are fixed by the limit $\lambda \to 0$, leading to the equations:

$$\hat{c}\left[1 - \lambda^2 + O\left(\lambda^4\right)\right] \approx \hat{a}\left[\lambda - \frac{\lambda^3}{6} + O\left(\lambda^5\right)\right] + \hat{b}\left[1 - \frac{\lambda^2}{2} + O\left(\lambda^4\right)\right], \qquad \text{(C.13)}$$

$$\hat{c}\left[1 - \lambda^2 + O\left(\lambda^4\right)\right] \approx -\hat{a}\left[\frac{1}{\lambda} - \frac{\lambda}{2} + O\left(\lambda^3\right)\right] + \hat{b}\left[1 - \frac{\lambda^2}{6} + O\left(\lambda^4\right)\right]. \qquad \text{(C.14)}$$

Keeping the lowest orders of λ only, the best approximative solution to these equations is given by $\hat{a} = 0, \hat{c} = \hat{b} = 1$. Therefore, the (regular) normalized eigenfunctions near the horizon and asymptotically should behave like

$$\varphi_\lambda(y) \propto \sqrt{\frac{2}{\pi}} \cdot \begin{cases} \mathrm{e}^{-\lambda^2 y}, & y \to 0\,, \\ \cos(\lambda y), & y \to \infty\,. \end{cases} \qquad \text{(C.15)}$$

C.2 Regularity

In $I_f E$ of (19) the behavior of the eigenfunctions (C.15) and of $F(\frac{\lambda^2}{4M^2 m^2})$ especially at small valus of λ seems to be sufficiently nice, so that the limit $m^2 \to 0$ can be performed and the divergence $\propto \ln m^2$ can be isolated:

$$I_f E \overset{m\to 0}{\to} \frac{8M^2}{\Box} \int_0^\infty \mathrm{d}\lambda \ln\left(\frac{\lambda^2}{4M^2 m^2}\right) \varphi_\lambda(y) \times \int_0^\infty \varphi_\lambda(y') \cdot E(y')\mathrm{d}y'\,. \qquad \text{(C.16)}$$

A necessary condition for that is the regularity of[14]

$$I_f^{\mathrm{reg}} E \propto \int_0^\infty \mathrm{d}\lambda \ln\lambda\, \varphi_\lambda(y) \int_0^\infty \varphi_\lambda(y') \cdot E(y')\mathrm{d}y'\,. \qquad \text{(C.17)}$$

The integral over y' can be split into the domains of the approximative solutions (C.15):

$$E_\lambda := \int_0^\infty \varphi_\lambda(y') \cdot E(y')\mathrm{d}y' \approx \int_0^1 \mathrm{e}^{-\lambda^2 y'} E(y')\mathrm{d}y' + \int_1^\infty \cos(\lambda y')E(y')\mathrm{d}y'\,. \qquad \text{(C.18)}$$

In SRG, where $E = \frac{2M}{r^3} \propto \frac{1}{(y+1)^3}$, we can approximate these integrals by the functions

$$\int_0^1 \frac{\mathrm{e}^{-\lambda^2 y'}}{(y'+1)^3}\mathrm{d}y' \approx \frac{1}{\lambda^2 + \frac{8}{3}}\,, \qquad \text{(C.19)}$$

$$\int_1^\infty \frac{\cos(\lambda y')}{(y'+1)^3}\mathrm{d}y' \approx \left(2\mathrm{e}^{-\lambda} - 1\right)\frac{\sin(\lambda)}{8\lambda}\,, \qquad \text{(C.20)}$$

which correctly reproduce the behavior of the integrals for small and large values of λ. For intermediate values the deviations are also small as we have checked graphically. The function

$$E_\lambda \approx \frac{1}{\lambda^2 + \frac{8}{3}} + \left(2\mathrm{e}^{-\lambda} - 1\right)\frac{\sin(\lambda)}{8\lambda} \qquad \text{(C.21)}$$

is regular for all values of λ, taking the value 1/2 at $\lambda = 0$ and behaving like $\sin\lambda/\lambda$ at infinity. Inserting (C.21) into (C.17) the behavior of $I_f^{\mathrm{reg}} E$ at either the horizon ($y \to 0$) or asymptotically ($y \to \infty$) can be investigated.

At the horizon only the part $\sin(\lambda)/\lambda$ of E_λ may cause problems in (C.17) for $\lambda \to \infty$. However, $\int_0^\infty \mathrm{d}\lambda(\ln\lambda\sin\lambda)/\lambda$ is finite, thus establishing regularity of (C.17) for $y \to 0$.

To show the asymptotic behavior of $I_f^{\mathrm{reg}} E$ is more tedious. First we consider the integral $\int_0^\infty \ln\lambda\cos(\lambda y)/(\lambda^2 + 8/3)\mathrm{d}\lambda$ from (C.19) in (C.17) in the limit $y \to \infty$. We split it at the point $\lambda = 1$. The first contribution is

$$\begin{aligned} \int_0^1 \frac{\ln\lambda\cos(\lambda y)}{\lambda^2 + \frac{8}{3}}\mathrm{d}\lambda &= \frac{1}{y}\int_0^y \frac{\ln\left(\frac{s}{y}\right)\cos s}{\frac{s}{y}^2 + \frac{8}{3}}\mathrm{d}s \\ &< \frac{3}{8y}\int_0^y \ln\left(\frac{s}{y}\right)\cos s\mathrm{d}s \\ &= \frac{3}{8y}\left[\ln\left(\frac{s}{y}\right)\sin s\Big|_0^y - \int_0^y \frac{\sin s}{s}\mathrm{d}s\right] \\ &= \frac{3\pi}{16\,y} + O(y^{-2})\,, \end{aligned} \qquad \text{(C.22)}$$

where a substitution $s = \lambda y$ has been performed. The second contribution by

[14] For the moment we neglect the action of $\Box^{-1}$ in I_f. Pulling it into the λ-integral would produce the typical IR problem of the two-dimensional Green function whose resolution shall not be discussed here (cf. the last paragraph of this appendix).

$$\begin{aligned}\Big|\int_1^\infty \frac{\ln\lambda\cos(\lambda y)}{\lambda^2+\frac{8}{3}}\mathrm{d}\lambda\Big| &= \frac{1}{y}\Big|\left\{\frac{\ln\lambda\sin(\lambda y)}{\lambda^2+\frac{8}{3}}\Big|_1^\infty\right. \\ &\quad -\int_1^\infty \sin(\lambda y) \\ &\quad \left.\times\left[\frac{1}{\lambda\left(\lambda^2+\frac{8}{3}\right)}-\frac{2\lambda\ln\lambda}{\left(\lambda^2+\frac{8}{3}\right)^2}\right]\mathrm{d}\lambda\right\}\Big| \\ &< \frac{1}{y}\left\{\Big|\int_1^\infty \frac{1}{\lambda\left(\lambda^2+\frac{8}{3}\right)}\mathrm{d}\lambda\Big|\right. \\ &\quad \left.+\Big|\int_0^1 \frac{2\lambda\ln\lambda}{\left(\lambda^2+\frac{8}{3}\right)^2}\mathrm{d}\lambda\Big|\right\} \\ &= \frac{3\ln(11/3)}{8\,y} \end{aligned} \tag{C.23}$$

is also at most of order y^{-1}. Next we consider the integral $\int_0^\infty \ln\lambda\cos(\lambda y)\left(2\mathrm{e}^{-\lambda}-1\right)\sin(\lambda)/(8\lambda)\mathrm{d}\lambda$. To obtain an upper bound it is sufficient to set $\left(2\mathrm{e}^{-\lambda}-1\right)\to 1$:

$$\begin{aligned}&\int_0^\infty \frac{\ln\lambda\cos(\lambda y)\sin\lambda}{\lambda} = \\ &\frac{1}{2}\int_0^\infty \frac{\ln\lambda\left\{\sin[\lambda(y+1)]-\sin[\lambda(y-1)]\right\}}{\lambda}\mathrm{d}\lambda = \\ &\frac{1}{2}\int_0^\infty \frac{\left\{\ln\left(\frac{s}{y+1}\right)\sin s-\ln\left(\frac{s}{y-1}\right)\sin s\right\}}{s}\mathrm{d}s = \\ &\frac{1}{2}\ln\left(\frac{y-1}{y+1}\right)\int_0^\infty \frac{\sin s}{s}\mathrm{d}s \overset{y\to\infty}{\to} 0-\frac{\pi}{2\,y}+O(y^{-2})\,. \end{aligned} \tag{C.24}$$

From the first to the second line we have performed a substution $s=\lambda(y\pm 1)$, respectively.

The present analysis can only be seen as some basic argument for the regularity of $\ln\Box E$. In the effective action one has to deal with terms $E\Box^{-1}\ln\Box E$, where the $\Box^{-1}$ can be made acting to the left. In the case of expectation values resulting from variations for E, however, one is forced to evaluate $I_f E$ as it stands. In this case an explicit analysis of the exact Green function (in terms of some expansion) seems inevitable.

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